

On Riemannian almost product manifolds with nonintegrable structure

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Abstract. The class of the Riemannian almost product manifolds with nonintegrable structure is considered. Some identities for curvature tensor as certain invariant tensors and quantities are obtained.

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Introduction

The systematic development of the theory of Riemannian almost product manifolds was started by K. Yano [8]. In [2] A. M. Naveira gives a classification of these manifolds with respect to the covariant differentiation of the almost product structure.

Such manifolds with zero trace of the almost product structure are considered in [1], [3]–[7]. Moreover, a classification is presented in [5], having in mind the results in [2]. In the classification in [5] the basic class \mathcal{W}_3 is only the class with nonintegrable structure.

In the present work the problems in the differential geometry of the manifolds of the class \mathcal{W}_3 are mainly considered.

1 Preliminaries

Let (M, P, g) be a Riemannian almost product manifold, i.e. a differentiable manifold M with a tensor field P of type $(1, 1)$ and a Riemannian metric g such that

$$P^2x = x, \quad g(Px, Py) = g(x, y) \quad (1)$$

for arbitrary x, y of the algebra $\mathfrak{X}(M)$ of the smooth vector fields on M . Obviously $g(Px, y) = g(x, Py)$.

Further x, y, z, w will stand for arbitrary elements of $\mathfrak{X}(M)$.

In this work we consider Riemannian almost product manifolds with $\text{tr}P = 0$. In this case (M, P, g) is an even-dimensional manifold.

If $\dim M = 2n$ then the associated metric \tilde{g} of g , determined by $\tilde{g}(x, y) = g(x, Py)$, is an indefinite metric of signature (n, n) . Since $\tilde{g}(Px, Py) = \tilde{g}(x, y)$, the manifold (M, P, \tilde{g}) is a pseudo-Riemannian almost product manifold. We say that (M, P, \tilde{g}) is an *associated manifold* of (M, P, g) .

The classification from [5] of Riemannian almost product manifolds is made with respect to the tensor field F of type $(0,3)$, defined by

$$F(x, y, z) = g((\nabla_x P)y, z), \quad (2)$$

where ∇ is the Levi-Civita connection of g . The tensor F has the following properties:

$$F(x, y, z) = F(x, z, y) = -F(x, Py, Pz), \quad F(x, y, Pz) = -F(x, Py, z). \quad (3)$$

The only class of Riemannian almost product manifolds with nonintegrable structure is the basic class \mathcal{W}_3 determined by the condition

$$\mathfrak{S}_{x,y,z} F(x, y, z) = 0, \quad (4)$$

where $\mathfrak{S}_{x,y,z}$ is the cyclic sum by x, y, z .

Further manifolds of the class \mathcal{W}_3 we call *Riemannian \mathcal{W}_3 -manifolds*.

The condition (4) is equivalent to

$$\mathfrak{S}_{x,y,z} F(Px, y, z) = 0. \quad (5)$$

In [5] the symmetric tensor field \bar{N} is defined by

$$\bar{N}(x, y) = (\nabla_x P)Py + (\nabla_{Px} P)y + (\nabla_y P)Px + (\nabla_{Py} P)x \quad (6)$$

and it has the properties

$$\bar{N}(Px, Py) = \bar{N}(x, y), \quad \bar{N}(Px, y) = \bar{N}(x, Py), \quad \bar{N}(x, Py) = -P\bar{N}(x, y).$$

It is proved that the condition (4) is equivalent to $\bar{N}(x, y) = 0$.

The class \mathcal{W}_0 , defined by the condition $F(x, y, z) = 0$, is contained in the other classes. This is the class of so-called *Riemannian P -manifolds*, i.e. differentiable even-dimensional manifolds (M, P, g) with Riemannian metric g and structure P , such that $g(Px, Py) = g(x, y)$, $P^2 = \text{id}$, $\text{tr} P = 0$, $\nabla P = 0$. Therefore the class \mathcal{W}_0 is an analogue of the class of Kählerian manifolds in the geometry of almost Hermitian manifolds.

The following property of the covariant derivation of F is valid

$$(\nabla_x F)(y, z, Pw) + (\nabla_x F)(y, Pz, w) = A(x, y, z, w), \quad (7)$$

where

$$A(x, y, z, w) = -g((\nabla_x P)z, (\nabla_y P)w) - g((\nabla_y P)z, (\nabla_x P)w). \quad (8)$$

As it is known the curvature tensor field R of a Riemannian manifold with metric g is determined by

$$R(x, y)z = \nabla_x \nabla_y z - \nabla_y \nabla_x z - \nabla_{[x, y]} z$$

and the corresponding tensor field of type $(0, 4)$ is defined as follows

$$R(x, y, z, w) = g(R(x, y)z, w).$$

Let (M, P, g) be a Riemannian almost product manifold and $\{e_i\}$ be a basis of the tangent space $T_p M$ at a point $p \in M$. Let the components of the inverse matrix of g with respect to $\{e_i\}$ be g^{ij} . If ρ and τ are the Ricci tensor and the scalar curvature, then ρ^* and τ^* , defined by $\rho^*(y, z) = g^{ij} R(e_i, y, z, P e_j)$ and $\tau^* = g^{ij} \rho^*(e_i, e_j)$, are called an *associated Ricci tensor* and an *associated scalar curvature*, respectively. We denote $\tau^{**} = g^{ij} g^{kl} R(e_i, e_k, P e_l, P e_j)$.

The Lie form θ associated to F is defined by $\theta(z) = g^{ij} F(e_i, e_j, z)$. For a Riemannian \mathcal{W}_3 -manifold it is valid $\theta(z) = 0$.

The square norm of ∇P is defined by $\|\nabla P\|^2 = g^{ij} g^{kl} g((\nabla_{e_i} P) e_k, (\nabla_{e_j} P) e_l)$.

2 A basic identity for the curvature tensor of Riemannian \mathcal{W}_3 -manifolds

Let (M, P, g) be a Riemannian \mathcal{W}_3 -manifold. According to (5) we have

$$\mathfrak{S}_{y, z, w} F(Py, z, w) = 0. \quad (9)$$

Then the covariant differentiations of (9) and (4) imply, respectively, the following equations

$$\mathfrak{S}_{y, z, w} (\nabla_x F)(Py, z, w) = \mathfrak{S}_{y, z, w} g((\nabla_x P)y, (\nabla_z P)w + (\nabla_w P)z), \quad (10)$$

$$\mathfrak{S}_{y, z, w} (\nabla_x F)(y, z, w) = 0. \quad (11)$$

Having in mind the Ricci identity

$$(\nabla_x \nabla_y P)z - (\nabla_y \nabla_x P)z = R(x, y)Pz - PR(x, y)z$$

and the properties (3) it follows immediately

$$(\nabla_x F)(y, z, w) - (\nabla_y F)(x, z, w) = R(x, y, Pz, w) - R(x, y, z, Pw). \quad (12)$$

Using (12) we get that the left-hand side of (10) has the form

$$\mathfrak{S}_{y,z,w} (\nabla_{Py} F)(x, z, w) + \mathfrak{S}_{y,z,w} \{R(x, Py, Pz, w) - R(x, Py, z, Pw)\}. \quad (13)$$

We apply (11) to each of the addends of the first cyclic sum in (13) and then we apply (10), (11), (12) in the obtained expression. After that we use the properties of R and we finally get the following identity

$$\begin{aligned} & \mathfrak{S}_{x,y,z} \{R(x, Py, Pz, w) - R(x, Py, z, Pw) \\ & \quad + R(Px, y, z, Pw) - R(Px, y, Pz, w)\} \\ & = \mathfrak{S}_{x,y,z} g((\nabla_x P)y + (\nabla_y P)x, (\nabla_z P)w + (\nabla_w P)z). \end{aligned} \quad (14)$$

In this way we prove the following

Theorem 2.1. *If (M, P, g) is a Riemannian \mathcal{W}_3 -manifold, then the curvature tensor R has the property (14).*

Having in mind the last theorem we prove the following

Corollary 2.2. *If (M, P, g) is a Riemannian \mathcal{W}_3 -manifold, then the following equations are valid*

$$\begin{aligned} & \rho(y, z) + \rho(Py, Pz) - \rho^*(Py, z) - \rho^*(y, Pz) = \\ & = g^{ij} g((\nabla_{e_i} P)y + (\nabla_y P)e_i, (\nabla_{e_j} P)z + (\nabla_z P)e_j), \end{aligned} \quad (15)$$

$$\|\nabla P\|^2 = -2g^{ij}g^{kl}g((\nabla_{e_i} P)e_k, (\nabla_{e_l} P)e_j), \quad (16)$$

$$\|\nabla P\|^2 = 2(\tau - \tau^{**}). \quad (17)$$

Proof. It is satisfied (14) for a Riemannian \mathcal{W}_3 -manifold hence we obtain (15). Having in mind (4) and the definition of $\|\nabla P\|^2$, we have (16). The equations (15) and (16) imply (17). \square

Remark 2.1. *If (M, P, g) is a Riemannian \mathcal{W}_3 -manifold with $\dim M \geq 4$ and a Kähler curvature tensor R , i.e. $R(x, y, Pz, Pw) = R(x, y, z, w)$, then $\tau^{**} = \tau$ and therefore $\|\nabla P\|^2 = 0$.*

3 Invariant bisectional curvature

Let (M, P, g) be a Riemannian almost product manifold and let α be a 2-plane in the tangent space at an arbitrary point of M . Then, it is known, the *sectional curvature* of α is defined by the following equation

$$k(x, y) = \frac{R(x, y, y, x)}{g(x, x)g(y, y) - g^2(x, y)},$$

where (x, y) is an arbitrary basis of α . If x is a noneigenvector of P , then the 2-plane $\alpha = (x, Px)$ is called an *invariant 2-plane*, because $P\alpha = \alpha$.

Let us define the quantity

$$h(x, y) = \frac{R(x, Px, y, Py)}{\sqrt{g^2(x, x) - g^2(x, Px)}\sqrt{g^2(y, y) - g^2(y, Py)}}, \quad (18)$$

where x, y are noneigenvectors of P .

It is known from [6] that an orthonormal adapted basis exists in every invariant 2-plane. Let this basis be (e_1, Pe_1) for $\alpha_1 = (x, Px)$. Then we have $x = \lambda e_1 + \mu Pe_1$, $Px = \lambda Pe_1 + \mu e_1$ ($\lambda, \mu \in \mathbb{R}$), hence $g(x, x) = \lambda^2 + \mu^2$, $g(x, Px) = 2\lambda\mu$. Therefore $g^2(x, x) - g^2(x, Px) = (\lambda^2 - \mu^2)^2 \geq 0$. The assumption $\lambda^2 = \mu^2$ (i.e. $\lambda = \pm\mu$) implies $x = \pm Px$, which is a contradiction to the condition x to be a noneigenvector of P . Hence we get $g^2(x, x) - g^2(x, Px) > 0$ and therefore the quantity $h(x, y)$ is defined correctly in (18).

Theorem 3.1. *The quantity $h(x, y)$ defined by (18) for a Riemannian almost product manifold (M, P, g) depends on the invariant 2-planes $\alpha_1 = (x, Px)$ and $\alpha_2 = (y, Py)$ only.*

Proof. Let $z \in \alpha_1$, $w \in \alpha_2$ be noneigenvectors of P . Then $z = \lambda_1 x + \mu_1 Px$, $w = \lambda_2 y + \mu_2 Py$ ($\lambda_1, \lambda_2, \mu_1, \mu_2 \in \mathbb{R}$), from where we have

$$\begin{aligned} g^2(z, z) - g^2(z, Pz) &= (\lambda_1^2 - \mu_1^2) (g^2(x, x) - g^2(x, Px)), \\ g^2(w, w) - g^2(w, Pw) &= (\lambda_2^2 - \mu_2^2) (g^2(y, y) - g^2(y, Py)), \\ R(z, Pz, w, Pw) &= (\lambda_1^2 - \mu_1^2) (\lambda_2^2 - \mu_2^2) R(x, Px, y, Py). \end{aligned}$$

Therefore $h(x, y) = h(z, w)$. □

The quantity $h(x, y)$ defined by (18) we call an *invariant bisectional curvature* of invariant 2-planes $\alpha_1 = (x, Px)$ and $\alpha_2 = (y, Py)$. In particular, if $x = y$, the 2-planes α_1 and α_2 coincide and then $h(x, x) = k(x, Px)$, i.e. $h(x, x)$ is the Riemannian sectional curvature of the invariant 2-plane (x, Px) .

Theorem 3.2. *A Riemannian almost product manifold (M, P, g) has a zero invariant bisectonal curvature if and only if the following condition is satisfied*

$$R(x, Py, Pz, w) - R(x, Py, z, Pw) + R(Px, y, z, Pw) - R(Px, y, Pz, w) = 0. \quad (19)$$

Proof. Let (M, P, g) be a Riemannian almost product manifold with $h(x, y) = 0$. Then $R(x, Px, y, Py) = 0$. In the last equation let $x + z$ and $y + w$ stand for x and y , respectively. Hence we obtain

$$R(x, Pz, y, Pw) + R(x, Pz, w, Py) + R(z, Px, y, Pw) + R(z, Px, w, Py) = 0.$$

Next we substitute $y \leftrightarrow z$ in the last equation and we get (19).

Now, let (19) be valid. There let x and z stand for y and w , respectively. Then we obtain $R(x, Px, z, Pz) = 0$, i.e. the manifold has a zero invariant bisectonal curvature. \square

Theorem 2.1, Theorem 2.2 and Theorem 3.2 imply immediately the following

Corollary 3.3. *If a Riemannian \mathcal{W}_3 -manifold (M, P, g) has a zero invariant bisectonal curvature then the following conditions are valid:*

$$\mathfrak{S}_{x,y,z} g((\nabla_x P)y + (\nabla_y P)x, (\nabla_z P)w + (\nabla_w P)z); \quad (20)$$

$$\tau^{**} = \tau; \quad (21)$$

$$\|\nabla P\|^2 = 0. \quad (22)$$

4 An associated pseudo-Riemannian almost product manifold

Let (M, P, g) be a Riemannian almost product manifold and (M, P, \tilde{g}) be the associated pseudo-Riemannian almost product manifold. We denote the Levi-Civita connection of \tilde{g} by $\tilde{\nabla}$. In [5] the tensor field Φ of type (1,2) is defined by

$$\Phi(x, y) = \tilde{\nabla}_x y - \nabla_x y$$

and the corresponding tensor field of type (0,3) by

$$\Phi(x, y, z) = g(\tilde{\nabla}_x y - \nabla_x y, z). \quad (23)$$

It is proved that

$$\Phi(x, y, z) = \frac{1}{2}(F(x, y, Pz) + F(y, Pz, x) - F(z, x, y)). \quad (24)$$

We have a classification of pseudo-Riemannian almost product manifolds (M, P, \tilde{g}) with respect to the tensor field \tilde{F} of type $(0,3)$, defined by $\tilde{F}(x, y, z) = \tilde{g} \left(\left(\tilde{\nabla}_x P \right) y, z \right)$. This classification is analogous to the one of Riemannian almost product manifolds (M, P, g) with respect to F . The class defined by $\mathfrak{S}_{x,y,z} \tilde{F}(x, y, z) = 0$ is the only class of nonintegrable structure P . This class will be denoted by \mathcal{W}_3 , too, and its manifolds will be called *pseudo-Riemannian \mathcal{W}_3 -manifolds*. The class \mathcal{W}_0 , defined by $\tilde{F}(x, y, z) = 0$ is contained in the other classes. This is the class of *pseudo-Riemannian P -manifolds*, i.e. differentiable $2n$ -dimensional manifolds (M, P, \tilde{g}) with pseudo-Riemannian metric \tilde{g} of signature (n, n) and a structure P such that

$$\tilde{g}(Px, Py) = \tilde{g}(x, y), \quad P^2 = \text{id}, \quad \text{tr} P = 0, \quad \tilde{\nabla} P = 0.$$

Theorem 4.1. *A Riemannian almost product manifold (M, P, g) is a \mathcal{W}_3 -manifold if and only if its associated pseudo-Riemannian almost product manifold (M, P, \tilde{g}) is also a \mathcal{W}_3 -manifold.*

Proof. Let (M, P, g) be a Riemannian \mathcal{W}_3 -manifold. Using (23), (24) and (4), we obtain immediately

$$\tilde{\nabla}_x y = \nabla_x y - (\nabla_x P) Py - (\nabla_y P) Px, \quad (25)$$

hence we have

$$\left(\tilde{\nabla}_x P \right) y = -(\nabla_{Py} P) Px - (\nabla_x P) y - (\nabla_y P) x. \quad (26)$$

Since for a Riemannian \mathcal{W}_3 -manifold (M, P, g) the tensor \bar{N} , defined by (6), vanishes, then (26) has the form

$$\left(\tilde{\nabla}_x P \right) y = (\nabla_{Px} P) Py. \quad (27)$$

According to properties $\tilde{g}(x, y) = g(x, Py) = \tilde{g}(Px, Py)$, (2) and (27), we get

$$\tilde{F}(x, y, z) = -F(Px, y, z), \quad (28)$$

from where

$$\mathfrak{S}_{x,y,z} \tilde{F}(x, y, z) = - \mathfrak{S}_{x,y,z} F(Px, y, z) = 0. \quad (29)$$

The equations (29) and (5) imply that (M, P, \tilde{g}) is also a \mathcal{W}_3 -manifold. Inversely, if (M, P, \tilde{g}) is a \mathcal{W}_3 -manifold, then, according to (29) and (5), we have that (M, P, g) is also \mathcal{W}_3 -manifold. \square

Remark 4.1. *The condition (28) implies that a Riemannian \mathcal{W}_3 -manifold (M, P, g) is a Riemannian P -manifold if and only if its associated manifold (M, P, \tilde{g}) is a pseudo-Riemannian P -manifold.*

Theorem 4.2. *Let R be the curvature tensor of the Levi-Civita connection ∇ for a Riemannian \mathcal{W}_3 -manifold (M, P, g) and let \tilde{R} be the curvature tensor of the Levi-Civita connection $\tilde{\nabla}$ for its associated manifold (M, P, \tilde{g}) . Then the following condition is valid*

$$\begin{aligned} \tilde{R}(x, y, z, w) = & R(x, y, z, Pw) - (\nabla_x F)(w, y, z) + (\nabla_y F)(w, x, z) \\ & + g((\nabla_y P)z + (\nabla_z P)y, (\nabla_x P)Pw + (\nabla_w P)Px) \\ & - g((\nabla_x P)z + (\nabla_z P)x, (\nabla_y P)Pw + (\nabla_w P)Py). \end{aligned} \quad (30)$$

Proof. It is known that at the transformation $\nabla \rightarrow \tilde{\nabla}$ it satisfies the following

$$\tilde{R}(x, y)z = R(x, y)z + Q(x, y)z, \quad (31)$$

where

$$Q(x, y)z = (\nabla_x T)(y, z) - (\nabla_y T)(x, z) + T(x, T(y, z)) - T(y, T(x, z)), \quad (32)$$

$$T(x, y) = -(\nabla_x P)Py - (\nabla_y P)Px. \quad (33)$$

The corresponding tensor \tilde{R} of type (0,4) is

$$\tilde{R}(x, y, z, w) = \tilde{g}(\tilde{R}(x, y)z, w).$$

Hence, according to (31), we have

$$\tilde{R}(x, y, z, w) = R(x, y, z, Pw) + g(Q(x, y)z, Pw). \quad (34)$$

Using (3), (4), (32) and (33) we establish after some transformations the following

$$\begin{aligned} g(Q(x, y)z, Pw) = & -(\nabla_x F)(w, y, z) + (\nabla_y F)(w, x, z) \\ & + g((\nabla_y P)z + (\nabla_z P)y, (\nabla_x P)Pw + (\nabla_w P)Px) \\ & - g((\nabla_x P)z + (\nabla_z P)x, (\nabla_y P)Pw + (\nabla_w P)Py). \end{aligned} \quad (35)$$

Then (34) and (35) imply (30). \square

5 Invariant tensors of the transformation $\nabla \rightarrow \tilde{\nabla}$

Let ∇ and $\tilde{\nabla}$ be the Levi-Civita connections of a Riemannian \mathcal{W}_3 -manifold (M, P, g) and its associated manifold (M, P, \tilde{g}) , respectively. An important problem is the finding of invariant tensors of the transformation $\nabla \rightarrow \tilde{\nabla}$.

We shall prove the following

Theorem 5.1. *Let ∇ and $\tilde{\nabla}$ be the Levi-Civita connections of a Riemannian \mathcal{W}_3 -manifold (M, P, g) and its associated manifold (M, P, \tilde{g}) , respectively. Then at the transformation $\nabla \rightarrow \tilde{\nabla}$ the tensors*

$$S(x, y) = \nabla_x y + \frac{1}{2}T(x, y), \quad L(x, y)z = R(x, y)z + \frac{1}{2}Q(x, y)z \quad (36)$$

are invariant, where $T(x, y)$ and $Q(x, y)z$ are determined by (33) and (32), respectively.

Proof. Having in mind $(\tilde{\nabla}_x P)y = \tilde{\nabla}_x Py - P\tilde{\nabla}_x y$ and (25), (33), we get

$$\tilde{T}(x, y) = -(\tilde{\nabla}_x P)Py - (\tilde{\nabla}_y P)Px = -T(x, y). \quad (37)$$

From (25) and (37) it follows

$$\tilde{\nabla}_x y + \frac{1}{2}\tilde{T}(x, y) = \nabla_x y + \frac{1}{2}T(x, y),$$

from where we have

$$\tilde{S}(x, y) = \tilde{\nabla}_x y + \frac{1}{2}\tilde{T}(x, y) = S(x, y).$$

This means that the tensor S is invariant with respect to the transformation $\nabla \rightarrow \tilde{\nabla}$.

At the transformation $\nabla \rightarrow \tilde{\nabla}$ the tensor Q , determined by (32), is transformed into the tensor \tilde{Q} , determined by

$$\tilde{Q}(x, y)z = (\tilde{\nabla}_x \tilde{T})(y, z) - (\tilde{\nabla}_y \tilde{T})(x, z) + \tilde{T}(x, \tilde{T}(y, z)) - \tilde{T}(y, \tilde{T}(x, z)). \quad (38)$$

Using (25), (37) and (38), we obtain

$$\tilde{Q}(x, y)z = -Q(x, y)z. \quad (39)$$

Then (31) and (39) imply immediately

$$\tilde{L}(x, y)z = \tilde{R}(x, y)z + \frac{1}{2}\tilde{Q}(x, y)z = L(x, y)z,$$

which means that the tensor L is also an invariant one with respect to the transformation $\nabla \rightarrow \tilde{\nabla}$. \square

In the next theorems some characteristics of Riemannian \mathcal{W}_3 -manifolds with vanishing invariant tensors S and L are given.

Theorem 5.2. *A Riemannian \mathcal{W}_3 -manifold with zero tensor S is a Riemannian P -manifold.*

Proof. Let (M, P, g) be a Riemannian \mathcal{W}_3 -manifold with $S = 0$. Then (36) and (33) imply

$$\nabla_x y = \frac{1}{2}((\nabla_x P)Py + (\nabla_y P)Px),$$

hence we obtain

$$(\nabla_y P)Px + (\nabla_{Py} P)x = 0. \quad (40)$$

In the last equation we substitute $x \leftrightarrow y$ and we have

$$(\nabla_x P)Py + (\nabla_{Px} P)y = 0. \quad (41)$$

Since (40) and (41), the Nijenhuis tensor of P has the form

$$N(x, y) = (\nabla_x P)Py + (\nabla_{Px} P)y - (\nabla_y P)Px - (\nabla_{Py} P)x = 0.$$

It is known from [5] that $N(x, y) = 0$ is a characteristic condition (M, P, g) to belong to the class $\mathcal{W}_1 \oplus \mathcal{W}_2$. Therefore $(M, P, g) \in (\mathcal{W}_1 \oplus \mathcal{W}_2) \cap \mathcal{W}_3 = \mathcal{W}_0$, i.e. (M, P, g) is a Riemannian P -manifold. \square

Theorem 5.3. *Let (M, P, g) be a Riemannian \mathcal{W}_3 -manifold with zero tensor L . Then the following identity is valid*

$$\begin{aligned} & 2(R(x, y, z, w) + R(x, Py, Pz, w) + \\ & + R(Px, Py, z, w) + R(Px, y, Pz, w)) = \\ & = 2g((\nabla_y P)Px + (\nabla_{Py} P)x, (\nabla_{Pw} P)z) \\ & + g((\nabla_y P)Pz + (\nabla_{Py} P)z, (\nabla_{Pw} P)x) \\ & + g((\nabla_z P)Px + (\nabla_{Pz} P)x, (\nabla_{Pw} P)y). \end{aligned} \quad (42)$$

Proof. Let (M, P, g) be a Riemannian \mathcal{W}_3 -manifold with $L = 0$. According to (36) we have

$$R(x, y)z + \frac{1}{2}Q(x, y)z = 0. \quad (43)$$

We denote

$$\begin{aligned} B(x, y, z, w) = & -g((\nabla_z P)y + (\nabla_y P)z, (\nabla_x P)w + (\nabla_{Pw} P)x) \\ & + g((\nabla_x P)z + (\nabla_z P)x, (\nabla_y P)w + (\nabla_{Pw} P)y). \end{aligned} \quad (44)$$

Then according to (43) and (35) we obtain

$$2R(x, y, z, w) = (\nabla_x F)(Pw, y, z) - (\nabla_y F)(Pw, x, z) + B(x, y, z, w). \quad (45)$$

In (45) we substitute Py and Pz for y and z , respectively, and then we add the obtained equation to (45). We get

$$\begin{aligned} 2(R(x, Py, Pz, w) + R(x, y, z, w)) &= \\ &= (\nabla_x F)(Pw, y, z) + (\nabla_x F)(Pw, Py, z) \\ &\quad - (\nabla_y F)(Pw, x, z) - (\nabla_{Py} F)(Pw, x, Pz) \\ &\quad + B(x, y, z, w) + B(x, Py, Pz, w). \end{aligned}$$

Hence, according to (7), we have

$$\begin{aligned} 2(R(x, Py, Pz, w) + R(x, y, z, w)) &= \\ &= A(x, Pw, Py, z) - A(Py, Pw, x, z) \\ &\quad + B(x, y, z, w) + B(x, Py, Pz, w) \\ &\quad - (\nabla_y F)(Pw, x, z) + (\nabla_{Py} F)(Pw, Px, z). \end{aligned} \tag{46}$$

Now, in (46) we substitute Px and Py for x and y , respectively, and then we add the obtained equation to (46). Having in mind $\tilde{N}(x, y) = 0$, (6), (7) and (44), we get (42). \square

Using the last theorem we prove the following

Corollary 5.4. *If (M, P, g) is a Riemannian \mathcal{W}_3 -manifold with zero tensor L , then*

$$\|\nabla P\|^2 = -8\tau, \tag{47}$$

$$\tau^{**} = 5\tau. \tag{48}$$

Proof. If (M, P, g) is a Riemannian \mathcal{W}_3 -manifold with $L = 0$, then we have (42). Hence we obtain immediately

$$\begin{aligned} 2(\rho(y, z) + \rho^*(Py, z) + \rho(Py, Pz) + \rho^*(y, Pz)) &= \\ &= 2g^{ij}g((\nabla_y P)e_i + (\nabla_{Py} P)Pe_i, (\nabla_{Pe_j} P)Pz) \\ &\quad + g^{ij}g((\nabla_z P)e_i + (\nabla_{Pz} P)Pe_i, (\nabla_{Pe_j} P)Py) \\ &\quad + g^{ij}g((\nabla_y P)z + (\nabla_{Py} P)Pz, (\nabla_{Pe_i} P)Pe_j). \end{aligned} \tag{49}$$

Since (M, P, g) is a Riemannian \mathcal{W}_3 -manifold, then $\theta = 0$ and (16) is valid. From (49), using (1), we obtain $4(\tau + \tau^{**}) = -3\|\nabla P\|^2$. Hence, according to (17), we get (47) and (48). \square

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